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$\mathscr{L}^*$-UNIPOTENT SEMIGROUPS

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The investigation of $\mathscr{L}^*$-unipotent semigroups is initiated. These are abundant semigroups in which the idempotents form a subsemigroup and every principal left *-ideal has a unique idempotent generator. The structure of super $\mathscr{L}^*$-unipotent semigroups is explained and used to obtain a structure theorem for $\mathscr{L}^*$-unipotent semigroups which are bands of cancellative monoids.

Introduction

Several authors (see for example [1, 2, 10, 12, 13, 14]) have studied *left* (*right*) *inverse* semigroups. These are regular semigroups in which each $\mathscr{L}$-class ($\mathscr{R}$-class) contains a unique idempotent or equivalently each principal left (right) ideal has a unique idempotent generator. Recently, abundant semigroups have been studied in several papers [3–7] and analogues of results in the regular theory have been obtained. It seems natural therefore, to look for analogues of results obtained in the left inverse case.

Recall that an abundant semigroup is one in which each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class contains an idempotent. Here two elements are $\mathscr{L}^*$-related ($\mathscr{R}^*$-related) in a semigroup if they are related by Green's relation $\mathscr{L}$ ($\mathscr{R}$) in some oversemigroup. The abundant analogue of left inverse is $\mathscr{L}^*$-unipotent where an abundant semigroup is *$\mathscr{L}^*$-unipotent* if its idempotents form a subsemigroup and each $\mathscr{L}^*$-class contains a unique idempotent. Our main results are inspired by those of Bailes [1], Edwards [2] and Venkatesan [13].

We start in Section 1 by giving a list of equivalent statements that characterise $\mathscr{L}^*$-unipotent semigroups and investigate a special congruence on such a semigroup. We say that a semigroup S is a *unipotent right adequate semigroup* if it contains a unique idempotent e and for $a, b, c \in S$ we have $ae = a$, and $ab = ac \Rightarrow eb = ec$. This notion is used in Section 2 in the study of a class of $\mathscr{L}^*$-unipotent semigroups.

We conclude the section by giving a set of equivalent statements that characterise $\mathscr{L}^*$-unipotent semigroups which are bands of unipotent right adequate semigroups. Super $\mathscr{L}^*$-unipotent semigroups are considered in Section 3 and we start with several characterisations which are used to investigate some of their properties. In Section

4 we study super $\mathscr{L}^*$-unipotent semigroups on which $\mathscr{H}^*$ is congruence and we give a structure theorem for such semigroups.

We use the notation and terminology of [9]. Other undefined terms can be found in the earlier paper [4].

1. Basic properties

A left ideal I of a semigroup S is a *left *-ideal* if it is a union of $\mathscr{L}^*$-classes. For an element a of S, $L^*(a)$ denotes the *principal* left *-ideal generated by a, that is, the unique smallest left *-ideal containing a. It is shown in [7] that $a\mathscr{L}^*b$ if and only if $L^*(a)=L^*(b)$. Furthermore, if S is abundant, then $L^*(a)=L^*(e)=Se$ for some idempotent e. If the idempotents of an abundant semigroup S form a subsemigroup, then S is said to be *quasi-adequate*. Clearly in a quasi-adequate semigroup we have $fef\mathscr{L}ef$ for any idempotents e,f. The following lemma is now clear:

Lemma 1.1. *Let S be a quasi-adequate semigroup and E its band of idempotents. Then the following statements are equivalent:*

(i) *S is $\mathscr{L}^*$-unipotent,*

(ii) *for any $a\in S$, there is a unique idempotent e in E such that $L^*(a)=Se$,*

(iii) *each principal left *-ideal has a unique idempotent generator,*

(iv) *$fef=ef$ for any $e,f\in E$.* □

The next result is similar to [6, Proposition 1.3] and [4, Proposition 1.3]:

Proposition 1.2. *Let S be a semigroup, E its set of idempotents and T the set of regular elements in S. Then the following conditions are equivalent:*

(i) *S is $\mathscr{L}^*$-unipotent,*

(ii) *T is a left inverse subsemigroup of S and E has non-empty intersection with each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class of S,*

(iii) *T is a left inverse subsemigroup of S and T has non-empty intersection with each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class of S,*

(iv) *$|R_a^*\cap E|\geq 1$ and $|L_a^*\cap E|=1$ for all $a\in S$, and the subsemigroup generated by E is regular.*

Proof. If (i) holds, then by definition each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class of S contains an idempotent. By [4, Proposition 1.3], T is an orthodox semigroup and by Lemma 1.1, $fef=ef$ for any $e,f\in E$. Now by [11, Theorem 1], T is a left inverse semigroup.

If (ii) holds, then since $E\subseteq T$, (iii) also holds.

If (iii) holds, then E is a subsemigroup of T. Let $a\in S$, $t\in L_a^*\cap T$ and t' be an inverse of t. Then by [8, result 9], $t't\mathscr{L}^*a$, that is, $|L_a^*\cap E|\geq 1$. Similarly, $|R_a^*\cap E|\geq 1$. $|L_a^*\cap E|=1$ follows from the fact that T is left inverse.

If (iv) holds, then to establish (i), we have only to show that E is a subsemigroup of S. Since the subsemigroup $\langle E\rangle$ generated by E is regular and for any $e, f\in E$, $e\mathscr{L}(\langle E\rangle)f$ if and only if $e\mathscr{L}(S)f$ [8], then $\langle E\rangle$ is left inverse. In particular, $\langle E\rangle$ is orthodox [1] so that $\langle E\rangle = E$. □

To add some more statements to the list we give as an analogue of [12, Theorem 1] and [2, Theorem 1.1] the following:

Proposition 1.3. *Let S be a quasi-adequate semigroup with band of idempotents E. The following statements are equivalent*:

(i) *S is $\mathscr{L}^*$-unipotent,*
(ii) *$eS\cap fS = efS$ for any $e, f\in E$,*
(iii) *$ef\mathscr{R}fe$ for any $e, f\in E$,*
(iv) *on E, Green's relations $\mathscr{R}$ and $\mathscr{J}$ coincide,*
(v) *$a^{\dagger}ea = ea$ for all $e\in E$, $a\in S$, $a^{\dagger}\in R_a^*\cap E$.*

Proof. We use the fact (Lemma 1.1) that S is $\mathscr{L}^*$-unipotent if and only if $fef = ef$ $(e, f\in E)$.

(i) ⇔ (ii). If $eS\cap fS = efS$, then $ef\in fS$ so that $fef = ef$. Now let $fef = ef$. Then $efS = fefS \subseteq fS$ and hence $efS\subseteq eS\subseteq fS$. On the other hand, if $x\in eS\cap fS$, then $x = ey = fz$ for some $y, z\in S$. Therefore $x = e(ey) = ex = efz \in efS$. It follows that $eS\cap fS = efS$.

(i) ⇒ (iii). Since $ef = fef = fe\cdot ef$ and $fe = efe = ef\cdot fe$ we have $ef\mathscr{R}fe$.

(iii) ⇒ (iv). Let $e, f\in E$ be $\mathscr{J}$-related in E. Then it is easy to see that $f\mathscr{R}fe$ and $e\mathscr{R}ef$. Hence by (iii), $e\mathscr{R}f$ and hence (iv) holds.

(iv) ⇒ (i). If $e\mathscr{L}f$, then (iv) gives $e\mathscr{H}f$ and consequently $e = f$.

(v) ⇔ (i). Clear. □

Consequently, we have the following corollary which is easily verified. Following [8] we denote Green's relation $\mathscr{J}$ on E by ε, and the $\mathscr{J}$-class containing $e\in E$ by $E(e)$.

Corollary 1.4. *Let S be an $\mathscr{L}^*$-unipotent semigroup with band of idempotents E, where $E = \bigcup_{\alpha\in\mathscr{Y}} E_\alpha$ and $\mathscr{Y}\simeq E/\varepsilon$. Then*:

(i) *There exists a one-to-one correspondence between E and the set of $\mathscr{L}^*$-classes in S.*

(ii) *There exists a one-to-one correspondence between $\mathscr{Y}$ and the set of R^*-classes in S such that $\alpha\in\mathscr{Y}$ corresponds to R_e^* if and only if $e\in E_\alpha$ for any $e\in E$.*

(iii) *$R_a^*\cap E = E(a^{\dagger})$ for any $a\in S$, $a^{\dagger}\in R_a^*\cap E$.* □

Corollary 1.5. *Let S be an $\mathscr{L}^*$-unipotent semigroup, E the set of its idempotents and $\theta: S\to T$ a good homomorphism of semigroups, then $S\theta$ is $\mathscr{L}^*$-unipotent.*

Proof. By [4, Corollary 1.7], $S\theta$ is a quasi-adequate semigroup whose band of idempotents is $\{e\theta: e\in E\}$. Then the result follows from Proposition 1.3 and the fact that θ is a homomorphism. □

It is now clear that the class of $\mathscr{L}^*$-unipotent semigroups includes left inverse semigroups and adequate semigroups, that is, abundant semigroups with commuting idempotents.

For an element a of an $\mathscr{L}^*$-unipotent semigroup S, the unique idempotent in L_a^* is denoted by a^*, and a typical idempotent in R_a^* is denoted by $a^\dagger$. We conclude this section with an observation about the minimum adequate congruence on S.

Proposition 1.6. *Let S be a quasi-adequate semigroup with band of idempotents E,*

$$\varrho' = \{(e,f) \in E \times E\colon fe = f \text{ and } ef = e\}$$

and ϱ the minimum good congruence on S containing ϱ'. Then ϱ is the minimum $\mathscr{L}^*$-unipotent good congruence on S.

Proof. By [4, Lemma 1.5 and Proposition 1.6] it follows that S/ϱ is a quasi-adequate semigroup with band of idempotens $\{e\varrho\colon e \in E\}$. Let $e\varrho, f\varrho$ be idempotents in S/ϱ such that $e\varrho \mathscr{L} f\varrho$, that is $ef\varrho = e\varrho$ and $fe\varrho = f\varrho$. Notice that $fe \cdot efe = fe$ and $efe \cdot fe = efe$ so that $(fe, efe) \in \varrho' \subseteq \varrho$. Then

$$e\varrho = ef\varrho = e\varrho f\varrho = e\varrho fe\varrho = efe\varrho = fe\varrho = f\varrho$$

and S/ϱ is $\mathscr{L}^*$-unipotent.

Let ϱ_1 be a good congruence on S such that S/ϱ_1 is $\mathscr{L}^*$-unipotent. If $(e,f) \in \varrho'$, then $f\varrho_1 \mathscr{L} e\varrho_1$ so that $f\varrho_1 = e\varrho_1$ and $\varrho' \subseteq \varrho_1$. Therefore $\varrho \subseteq \varrho_1$. □

If S is $\mathscr{L}^*$-unipotent,

$$\gamma' = \{(e,f) \in E \times E\colon ef = f,\ fe = e\},$$

and γ is the minimum good congruence on S containing γ', then by a similar argument to that of the above we have that γ is the minimum adequate good congruence on S. Now we proceed to find an explicit formula for γ on a class of $\mathscr{L}^*$-unipotent semigroups.

Recall from [4] that δ is defined on S by: $a\delta b$ if and only if $E(a^\dagger)aE(a^*) = E(b^\dagger)bE(b^*)$ for any $a, b \in S$.

By [4, Corollary 2.4(2)], $a\delta b$ if and only if $a = ebf$, $b = gah$, where $e \in E(b^\dagger)$, $f \in E(b^*)$, $g \in E(a^\dagger)$, $h \in E(a^*)$.

From [4, Corollary 2.3] and Corollary 1.4 it follows that

$$R_a^* \cap E = E(a^\dagger) = E(b^\dagger) = R_b^* \cap E$$

so that $a\mathscr{R}^*b$, and by [4, Lemma 2.2], $e\mathscr{R}^*a$, $f\mathscr{L}^*a$, $g\mathscr{R}^*b$, $h\mathscr{L}^*b$. Therefore

$$a\delta b \quad \text{if and only if} \quad a = ba^*,\ b = ab^*.$$

which is an analogue of [10, Lemma 2.3].

Note that for any $a, b \in S$, if $a = be$ for some $e \in E$, then $a^* = b^*e$ and $ba^* = bb^*e = be = a$. Therefore

$$a = ba^* \quad \text{if and only if} \quad a = be \text{ for some } e \text{ in } E.$$

This generalises [10, Lemma 2.1]. Further, it is now clear that δ is a left congruence.

Now suppose S is also idempotent-connected and $\theta : \langle a^\dagger \rangle \to \langle a^* \rangle$ is a connecting isomorphism for an element a in S. Let $e \in E$. Then $ea^\dagger = a^\dagger ea^\dagger \in \langle a^\dagger \rangle$ and $ea = ea^\dagger a = a(ea^\dagger)\theta$. From above we get $ea = a(ea)^*$. It follows that δ is congruence on any idempotent-connected $\mathscr{L}$*-unipotent semigroup S. This gives a partial answer to the question which was asked in [4, Section 3]. In this case, by [4, Proposition 2.6], δ is the minimum adequate good congruence on S and hence $\delta = \gamma$.

2. Unipotent right-adequate semigroups

A semigroup S is called *unipotent right adequate* if S contains a unique idempotent and $\mathscr{L}^*$ is the universal relation on S. A unipotent left adequate semigroup is defined dually. Clearly, T is a cancellative monoid if and only if T is both unipotent right and left adequate. Any left (right) cancellative monoid is a unipotent right (left) adequate semigroup, but a unipotent right adequate semigroup need not to be left cancellative as the following example from [6] demonstrates:

Example 2.1. Let A be the infinite cyclic semigroup with generator a and let B be the infinite cyclic monoid with generator b and identity e. Let $S = A \cup B$ and define a product on S which extends those on A and B by putting

$$a^m b^n = a^{m+n}, \qquad b^n a^m = b^{n+m}$$

for integers $m > 0$ and $n \geq 0$ where $b^0 = e$.

It is straightforward to check that S is a semigroup with just one idempotent e and consists of one $\mathscr{L}^*$-class. But S is not left cancellative since for example, $ab = a^2$.

Let S be an $\mathscr{L}^*$-unipotent semigroup with set of idempotents E. Define μ_L on S by

$$a\mu_L b \quad \text{if and only if} \quad (ea)^* = (eb)^* \text{ for any } e \in E.$$

The same argument as that in [3] shows that μ_L is the largest congruence in $\mathscr{L}^*$. The following lemma is an immediate consequence of this fact:

Lemma 2.2. *$e\mu_L$ is a unipotent right adequate semigroup for any $e \in E$.* □

As in [2] we define the *left neutraliser* $E\eta_L$ of E in S by

$$E\eta_L = \{x \in S\colon (ex)^* = ex^* \text{ for any } e \in E\}.$$

Lemma 2.3. $E\eta_L = \bigcup_{e \in E} e\mu_L$.

Proof. Clearly $x \in E\eta_L$ implies $x\mu_L x^*$. If $x \in f\mu_L$ for some $f \in E$, then $f = x^*$ so that $(ex)^* = (ex^*)^* = ex^*$ for all $e \in E$ and hence $x \in E\eta_L$. □

The following corollary is an easy consequence of Lemmas 2.2 and 2.3:

Corollary 2.4. *$E\eta_L$ is a band of unipotent right adequate semigroups.* □

Proposition 2.5. *The following statements are equivalent:*
(i) *each μ_L-class contains an idempotent,*
(ii) $\mathscr{L}^* = \mu_{L'}$,
(iii) *$\mathscr{L}^*$ is a congruence,*
(iv) $E\eta_L = S$,
(v) *S is a band of unipotent right adequate semigroups.*

Proof. If each μ_L-class contains an idempotent, then $a\mu_L a^*$ for any $a \in S$. Let $a\mathscr{L}^* b$, then $a\mu_L a^* = b^*\mu_L b$ and $\mu_L = \mathscr{L}^*$.

Now it is clear that (i), (ii) and (iii) are equivalent.

If (iii) holds, then $ex\mathscr{L}^*ex^*$, that is, $(ex)^* = ex^*$ for any $e \in E$, $x \in S$. Therefore $S = E\eta_L$.

If (iv) holds, then (v) holds by Corollary 2.4.

If (v) holds, let $S = \bigcup_{\alpha \in \mathscr{I}} T_\alpha$ be a band of unipotent right adequate semigroups. Let e_α be the idempotent in T_α. If $a \in T_\alpha$, $e_\gamma a, e_\gamma e_\alpha \in T_{\gamma\alpha}$. Hence, $(e_\gamma a)^* = (e_\gamma e_\alpha)^*$. So, $a\mu_L e_\alpha$ and thus, $e_\alpha \in a\mu_L$. Hence, condition (i) holds. □

Corollary 2.6. *If $S = E\eta_L$, then $S/\mu_L \simeq E$.*

Proof. Define $\theta : S \to E$ by $x\theta = x^*$. It is clear that θ is a map of S onto E. Since $S = E\eta_L$, we have for any $a, b \in S$,

$$(ab)\theta = (ab)^* = (a^*b)^* = a^*b^* = a\theta b\theta$$

so that θ is a homomorphism.

Further, if $a\theta = b\theta$, then $a\mathscr{L}^*b$ and so by Proposition 2.5, $a\mu_L b$. Clearly $a\mu_L b$ implies $a\theta = b\theta$. Therefore $\ker\theta = \mu_L$ and $S/\mu_L \simeq E$. □

In contrast to the left inverse semigroup case (see [2, Theorem 3.3] or [13, Theorem 7]) the converse of Corollary 2.6 does not hold as one can see from the following example:

Example 2.7. Let $T = (S^* \times R) \cup \{1\}$ where S^* is the dual of the semigroup S in Example 2.1 and $R = \{e_i : i \in N\}$ is a right zero semigroup. Clearly the idempotents of T form a subsemigroup

$$E = \{(e, e_i) : i \in N\} \cup \{1\}.$$

Using the notation of Example 2.1, it is readily verified that the R^*-classes of T are $\{1\}$ and $S^*\times R$ and the $\mathscr{L}^*$-classes of T are $(A\times R)\cup\{1\}$ and for each $i\in N$, $B\times\{e_i\}$, so that T is an $\mathscr{L}^*$-unipotent semigroup. Each $B\times\{e_i\}$ is also a μ_L-class but $(A\times R)\cup\{1\}$ splits into two μ_L-classes, $A\times R$ and $\{1\}$. Thus in view of Proposition 2.5, $T\neq E\eta_L$.

However, the map $\theta: T\to E$ defined by

$$1\theta=1, \qquad (a^m,e_i)\theta=(e,e_1), \qquad (b^m,e_i)\theta=(e,e_{i+1})$$

is easily seen to be a surjective homomorphism with $\ker\theta=\mu_L$. Thus $T/\mu_L\simeq E$.

3. Super $\mathscr{L}^*$-unipotent semigroups

A superabundant semigroup (see [7]) is an abundant semigroup in which every $\mathscr{H}^*$-class contains an idempotent. We say that a superabundant $\mathscr{L}^*$-unipotent semigroup is *super $\mathscr{L}^*$-unipotent*. In view of [7, Lemma 1.12], such a semigroup is a disjoint union of cancellative monoids. In our theory, we regard super $\mathscr{L}^*$-unipotent semigroups as the analogue of left inverse semigroups which are unions of groups. In this section and the subsequent one we extend some results from [1] and [13] on union of groups left inverse semigroups to super $\mathscr{L}^*$-unipotent semigroups. We start with the following lemma which follows immediately from the definition:

Lemma 3.1. *Let S be an $\mathscr{L}^*$-unipotent semigroup, then the following statements are equivalent*:

(i) *S is superabundant*,

(ii) $\mathscr{L}^*\subseteq\mathscr{R}^*$,

(iii) $\mathscr{D}^*=\mathscr{R}^*$,

(iv) $\mathscr{L}^*=\mathscr{H}^*$. □

Lemma 3.2. *Let S be an $\mathscr{L}^*$-unipotent semigroup. If e, f are $\mathscr{R}$-related idempotents in S and $a\in H_e^*$, then $af\in H_f^*$.*

Proof. Under the hypothesis, it is routine to check that $af\mathscr{L}^*f$ and $af\mathscr{R}^*f$ giving $af\in H_f^*$. □

Let S be an $\mathscr{L}^*$-unipotent semigroup and E be the set of its idempotents. Let $E=\bigcup_{\alpha\in\mathscr{Y}}E_\alpha$ be the maximal semilattice decomposition of E. Define $\phi: S\to\mathscr{Y}$ by $a\phi=\alpha$ if $a^*\in E_\alpha$. ϕ is a well-defined map on S onto $\mathscr{Y}$. If S is a superabundant, then by Lemma 3.1 $ab\mathscr{L}^*a^*b\mathscr{R}^*a^*b^*$ so that $(ab)^*\mathscr{R}^*a^*b^*$ for any $a,b\in S$. But by Corollary 1.4, $R_a^*\cap E=E(a^\dagger)$ and it follows that ϕ is a homomorphism. Moreover, if $a\mathscr{R}^*b$, then $a\phi=b\phi$; in particular, ϕ is good. Note that $\alpha\phi^{-1}=S_\alpha$ is a subsemigroup of S whose set of idempotents is E_α, $\mathscr{L}^*(S_\alpha)\subseteq\mathscr{L}^*(S)$, $\mathscr{R}^*(S_\alpha)\subseteq\mathscr{R}^*(S)$ and $S_\alpha S_\beta\subseteq S_{\alpha\beta}$. By [7, Proposition 6.5] and Lemma 3.1, $\mathscr{J}^*=\mathscr{D}^*=\mathscr{R}^*$ on S and for any $a,b\in S$ $a\mathscr{R}^*b$ implies $a,b\in S_\alpha$ for some $\alpha\in\mathscr{Y}$. On the other hand, if $a,b\in S_\alpha$

then $a\phi = b\phi$, that is, $a^*, b^* \in E_\alpha$ and by Corollary 1.4 and Lemma 3.1, $a\mathcal{R}^*b$. We abstract the notion of S_α as follows:

Proposition 3.3. *Let S be an $\mathcal{L}^*$-unipotent semigroup with band of idempotents E and $E = \bigcup_{\alpha \in \mathcal{Y}} E_\alpha$ be the maximal semilattice decomposition of E. For each $\alpha \in \mathcal{Y}$ define*

$$S_\alpha = \{x \in S\colon x^\dagger, x^* \in E_\alpha\}.$$

Then

(i) *S_α is the maximal abundant subsemigroup of S which contains E_α as its set of idempotents such that $\mathcal{L}^*(S_\alpha) \subseteq \mathcal{L}^*(S)$ and $\mathcal{R}^*(S_\alpha) \subseteq \mathcal{R}^*(S)$,*

(ii) *$S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$,*

(iii) *$\mathcal{R}^*$ is the universal relation on S_α.* □

Recall from [7] that a semigroup S without zero is called *completely $\mathcal{J}^*$-simple* if S is a primitive abundant semigroup whose idempotents generate a regular subsemigroup of S. Now we are in a position to give an analogue of [1, Theorem 2.7] as follows:

Proposition 3.4. *Under the hypothesis of Proposition* 3.3, *the following statements are equivalent*:

(i) *S is superabundant,*

(ii) *$S_\alpha = R_x^*(S)$ $(x \in S_\alpha)$ for each $\alpha \in \mathcal{Y}$,*

(iii) *$S = \bigcup_{\alpha \in \mathcal{Y}} S_\alpha$,*

(iv) *S is a semilattice of S_α's $(\alpha \in \mathcal{Y})$,*

(v) *S is a semilattice $\mathcal{X}$ of completely $\mathcal{J}^*$-simple semigroups S_α $(\alpha \in \mathcal{X})$ such that for $\alpha \in \mathcal{X}$ and $a \in S_\alpha$, $L_a^*(S) = L_a^*(S_\alpha)$, $R_a^*(S) = R_a^*(S_\alpha)$.*

Proof. (i) ⇒ (ii). By Proposition 3.3, $S_\alpha \subseteq R_x^*(S)$ for any element x of S_α. Let $y \in S$ be such that $y\mathcal{R}^*x$. By Lemma 3.1 and Corollary 1.4, y^*, $y^\dagger \in R_x^* \cap E = E_\alpha$. Thus $y \in S_\alpha$ and $S_\alpha = R_x^*(S)$.

(ii) ⇒ (iv). It is clear that S is a disjoint union of the S_α's $(\alpha \in \mathcal{Y})$. If $x \in S_\alpha$, $y \in S_\beta$, then $xy \in S_\alpha S_\beta$ and $x^*y^* \in E_\alpha E_\beta \subseteq E_{\alpha\beta}$. But $xy \in S$, say, $xy \in S_\gamma$ for some $\gamma \in \mathcal{Y}$. Since $y \in S_\beta$, then $y\mathcal{R}^*y^*$ so that $xy\mathcal{R}^*xy^*$, that is, $xy^* \in S_\gamma$ and $(xy^*) = x^*y^*$. Thus $x^*y^* \in E_\gamma$ and $\alpha\beta = \gamma$. Therefore $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

(iv) ⇒ (iii). Trivial.

(iii) ⇒ (i). Let $a \in S$, say, $a \in S_\alpha$. Then $a^\dagger, a^* \in E_\alpha$ and so $a^\dagger\mathcal{R}^*a^*$ by Corollary 1.4. Therefore $a\mathcal{R}^*a^*$ and $a\mathcal{H}^*a^*$. Hence S is superabundant.

(i) ⇔ (v). [7, Theorem 6.8]. □

Corollary 3.5. *If S is super $\mathcal{L}^*$-unipotent, then $\mathcal{R}^*$ is a congruence on S.*

Proof. By Proposition 3.4, S is a semilattice of the S_α's. Let $a, b, c \in S$ be such that $a\mathcal{R}^*b$, say, $R_a^*(S) = S_\alpha$ and $c \in S_\beta$, then $ac, bc \in S_{\alpha\beta}$, that is, $ac\mathcal{R}^*bc$ and $\mathcal{R}^*$ is a congruence. □

We now describe the algebraic structure of the subsemigroups S_α.

Proposition 3.6. *Let S be a super $\mathscr{L}^*$-unipotent semigroup with band of idempotents E and let S_α be as defined in Proposition 3.3. Then $S_\alpha \simeq M_\alpha \times E_\alpha$ for some cancellative monoid M_α.*

Proof. For any $e \in E_\alpha$, H_e^* is a cancellative monoid by [7, Lemma 1.12]. Let $M_\alpha = H_e^*$. Define $\psi : M_\alpha \times E_\alpha \to S_\alpha$ by $(a,f)\psi = af$. It is clear from Lemma 3.2 and the definition of S_α that $af \in S_\alpha$. If $a, b \in M_\alpha$, $f, g \in E_\alpha$ and $af = bg$, then

$$a = ae = afe = bge = be = b$$

and by Lemma 3.2, $f\mathscr{H}^*af = bg\mathscr{H}^*g$ so that $f = g$. Thus ψ is one-one. If $s \in S_\alpha$, then $s^* \in E_\alpha$ so that $s^*\mathscr{R}e$. Hence $(se, s^*) \in M_\alpha \times E_\alpha$ and

$$(se, s^*)\psi = ses^* = ss^* = s.$$

Hence ψ is onto. Note that

$$\begin{aligned}(a,f)\psi(b,g)\psi &= afbg \\ &= abg \quad (f, b \in S_\alpha \Rightarrow b\mathscr{R}^*f) \\ &= (ab, g)\psi \\ &= (ab, fg)\psi \quad (f\mathscr{R}g) \\ &= ((a,f)(b,g))\psi.\end{aligned}$$

Hence ψ is an isomorphism. □

4. Bands of cancellative monoids

The $\mathscr{L}^*$-unipotent semigroups which are bands of cancellative monoids will be investigated in this section. The main result is a structure theorem for such semigroups. We start with

Lemma 4.1. *A semigroups S is a band of cancellative monoids if and only if S is a superabundant and $\mathscr{H}^*$ is a congruence.*

Proof. Let $S = \bigcup_{\alpha \in \mathscr{X}} T_\alpha$ be a band of cancellative monoids T_α ($\alpha \in \mathscr{X}$), and let e_α be the identity of T_α. Let $a \in T_\alpha$, $s \in T_\beta$, $t \in T_\gamma$ be such that $as = at$. Then $\alpha\beta = \alpha\gamma = \delta$, say, and $e_\alpha s, e_\alpha t \in T_\delta$. Hence $ae_\delta e_\alpha s = ae_\delta e_\alpha t$ and since T_δ is cancellative, we have $e_\alpha s = e_\alpha t$. Thus $a\mathscr{L}^*e_\alpha$. Similarly $a\mathscr{R}^*e_\alpha$ and so S is superabundant. It is easy to see that $T_\alpha = H_a^*$ for any $a \in T_\alpha$ and it follows that $\mathscr{H}^*$ is a congruence.

On the other hand if S is superabundant, then each $\mathscr{H}^*$-class is a cancellative monoid and $x\mathscr{H}^*x^2$ for any $x \in S$. If $\mathscr{H}^*$ is a congruence, then $S/\mathscr{H}^*$ is a band and the natural homomorphism $\phi: S \to S/\mathscr{H}^*$ defined by $x\phi = H_x^*$ shows that S is a band of the $\mathscr{H}^*$-classes. □

Now let S be an $\mathscr{L}^*$-unipotent semigroup with band of idempotents E. We define $\mu = \mu_L \cap \mu_R$ where μ_L is as in Section 2 and

$$a\mu_R b \quad \text{if and only if} \quad ae\mathscr{R}^*be \text{ for any } e \in E.$$

We remark that by [3, Proposition 2.1], μ_L, μ_R, μ are the largest congruences contained in $\mathscr{L}^*$, $\mathscr{R}^*$ and $\mathscr{H}^*$ respectively. Thus for $e \in E$, $e\mu$ is contained in H_e^* and is easily seen to be cancellative submonoid of H_e^*.

We also define $E\eta = E\eta_L \cap E\eta_R$ where $E\eta_L$ is as defined in Section 2 and

$$E\eta_R = \{x \in S\colon x\mu_R \cap E \neq \emptyset\}.$$

It is easy to see that $E \subseteq E\eta$. We shall say that E is *neutral* in S if $S = E\eta$.

Lemma 4.2. $E\eta = \bigcup_{e\in E} e\mu$ *is a right regular band of cancellative monoids.*

Proof. By Lemma 1.1, $fef = ef$ for e, f in E, that is, E is a right regular band. If $x \in E\eta$, then $x \in E\eta_L$, and $(ex)^* = ex^* = (ex^*)^*$ for any $e \in E$. Therefore $x\mu_L x^*$. Also $x \in E\eta_R$ and so there exists an idempotent f in R_x^* such that $xe\mathscr{R}^*fe$ for any $e \in E$. In particular, $x = xx^*\mathscr{R}^*fx^*$ and $f\mathscr{R}fx^*$. Thus $fx^*f = f$. Since S is $\mathscr{L}^*$-unipotent, we have $x^*f = f$ and it follows that

$$x^*efe = x^*fe = fe$$

and $fex^*e = fx^*e = (fx)^*e = x^*e$ using the facts that $x \in E\eta_L$ and that $f\mathscr{R}^*x$.

Hence $x^*e\mathscr{R}fe\mathscr{R}^*xe$ so that $x\mu_R x^*$. Therefore $x\mu x^*$ and $x \in \bigcup_{e\in E} e\mu$. Now let $x \in f\mu$ for some $f \in E$. Then $x^* = f$ and $(ex)^* = ex^*$ for any $e \in E$. Therefore $x \in E\eta_L$. Also $f \in x\mu_R \cap E$. Thus $x \in E\eta_R$ and so $x \in E\eta$.

Hence $E\eta = \bigcup_{e\in E} e\mu$ and the result follows from the fact that μ is a congruence and $e\eta$ is a cancellative monoid for any $e \in E$. □

Proposition 4.3. *The following statements are equivalent:*

(i) $S/\mu \simeq E$ *and* $(a^*\mu)\mathscr{L}^*(S/\mu)\,(a\mu)$ *for any* $a \in S$,

(ii) E *is neutral in* S,

(iii) *each* μ*-class contains an idempotent,*

(iv) $a^*\mathscr{R}^*a$ *for any* $a \in S$ *and* $\mathscr{L}^*$ *is a congruence,*

(v) $\mathscr{H}^*$ *is a congruence and each* $\mathscr{H}^*$*-class contains an idempotent,*

(vi) S *is a band of cancellative monoids.*

Proof. If (i) holds, then S/μ is a right regular band, so that $\mathscr{L}^*$ is trivial on S/μ. Therefore $a\mu a^*$ for any $a \in S$ and $(ea)^* = (ea^*)^* = ea^*$ for any $e \in E$, that is, $a \in E\eta_L$. Also $a^*\mathscr{R}^*a$ and $ae\mathscr{R}^*a^*e$ for any $e \in E$ so that $a \in E\eta_R$. Therefore $a \in E\eta$ and $S = E\eta$.

That (iii) follows from (ii) is a consequence of Lemma 4.2.

If (iii) holds, then $a\mu a^*$ for any $a\in S$. Let $a,b\in S$ be such that $a\mathscr{L}^*b$. Then $a^*=b^*$ and $a\mu b$. Therefore $\mathscr{L}^*=\mu$ and $\mathscr{L}^*\subseteq\mathscr{H}^*\subseteq\mathscr{R}^*$ so that $a^*\mathscr{R}^*a$ for any $a\in S$ and (iv) holds.

If (iv) holds, then for any $a\in S$, $a^*\mathscr{R}^*a$, that is, $a\mathscr{H}^*a^*$. Therefore S is a super $\mathscr{L}^*$-unipotent and by Lemma 3.1, $\mathscr{H}^*=\mathscr{L}^*$, that is $\mathscr{H}^*$ is a congruence and (v) holds.

From Lemma 4.1 we have that (v) is equivalent to (vi).

If (v) holds, then $\mathscr{L}^*=\mathscr{H}^*=\mu$ and clearly $a^*\mu a$ for any $a\in S$. Define $\phi:S\to E$ by $a\phi=a^*$. It is easy to see that ϕ is a well-defined map of S onto E. Since $\mathscr{L}^*$ is a congruence, we have for any $a,b\in S$, $ab\mathscr{L}^*a^*b^*$, that is, $(ab)^*=a^*b^*$ and ϕ is a homomorphism. Note that

$$a\mu b \quad\Leftrightarrow\quad a\mathscr{L}^*b \quad\Leftrightarrow\quad a^*=b^*.$$

Thus $\ker\phi=\mu$, $S/\mu\simeq E$ and (i) holds. □

Notice that in Example 2.7 the $\mathscr{R}^*$-classes of T are $\{1\}$ and $T\backslash\{1\}$. Hence T is $\mathscr{L}^*$-unipotent and $\mu_L=\mu$ so that $T/\mu\simeq E$. But $\mathscr{L}^*$ is not a congruence and so by Proposition 4.3, T is not a band of cancellative monoids. Therefore the statement $S/\mu\simeq E$ in Proposition 4.3 is not enough by itself to give (ii) to (v). This is in contrast to the situation in left inverse semigroups (see [13]).

Let S be an $\mathscr{L}^*$-unipotent semigroup with band of idempotents E. Recall from the remark at the end of Section 1 that the relation δ defined there is a left congruence. Now if S is also a band of cancellative monoids, then by Proposition 4.3, S is super $\mathscr{L}^*$-unipotent and $\mathscr{H}^*=\mathscr{L}^*$ is a congruence. Let $a,b,c\in S$ with $a\delta b$. Since $ac\mathscr{H}^*a^*c^*$, $bc\mathscr{H}^*b^*c^*$ and $ca^*c^*\mathscr{H}^*c^*a^*c^*=a^*c^*$, we have

$$ac=ba^*c^*ca^*c^*=bc(a^*c^*)=bc(ac)^*.$$

Similarly, $bc=ac(bc)^*$, so that $ac\delta bc$ and δ is a congruence. This answers, for the class of $\mathscr{L}^*$-unipotent semigroups which are bands of cancellative monoids, the question posed in [4].

Lemma 4.4. *Let S be an $\mathscr{L}^*$-unipotent semigroup and E be the set of its idempotents. If S is a band of cancellative monoids, then S/δ is a semilattice of cancellative monoids.*

Proof. Since δ is a congruence on S, it is the minimum adequate good congruence on S by [4, Proposition 2.6]. In particular, S/δ is an adequate semigroup. Since each $\mathscr{H}^*$-class in S contains an idempotent and δ is good congruence, each $\mathscr{H}^*$-class in S/δ contains an idempotent. Now the result follows by [6, Proposition 2.9]. □

Let S_1,S_2 be semigroups. Assume there exist a semilattice $\mathscr{Y}$ and semigroup homomorphisms θ_i of S_i onto $\mathscr{Y}$ $(i=1,2)$. Then the set

$$P = [(s_1, s_2) \in S_1 \times S_2 \colon s_1\theta_1 = s_2\theta_2\}$$

is a subdirect product of S_1 and S_2 called a *spined product* of S_1 and S_2 (relative to the homomorphisms θ_1, θ_2).

Inspired by [11, Theorem 3.2] we have the following proposition:

Proposition 4.5. *Let S be an $\mathscr{L}^*$-unipotent semigroup with band of idempotents E. If S is a band of cancellative monoids, then S is a spined product of E and a semilattice of cancellative monoids.*

Proof. Recall from [9] that $\varepsilon^\natural : E \to E/\varepsilon$; $e\varepsilon^\natural = E(e)$ is a homomorphism and E/ε is the maximum semilattice homomorphic image of E.

Define $\eta : S/\delta \to E/\varepsilon$ by $(a\delta)\eta = E(a^*)$. If $a\delta b$, then $a = ba^*$, $b = ab^*$ so that $a^* = b^*a^*$, $b^* = a^*b^*$, that is, $a^*\mathscr{R}b^*$. In particular, $E(a^*) = E(b^*)$ so that η is well-defined. Clearly η is onto and since $(ab)^* = a^*b^*$ for any $a, b \in S$ it is easy to see that η is a homomorphism. Therefore,

$$P = \{(e, a\delta) \in E \times S/\delta \colon E(e) = (a\delta)\eta\}$$

is a spined product of E and S/δ which is a semilattice of cancellative monoids by Lemma 4.4.

It is clear that $\lambda : S \to P$ defined by $a\lambda = (a^*, a\delta)$ is a homomorphism. If $(e, a\delta) \in P$, then $E(e) = (a\delta)\eta = E(a^*)$ so that by Corollary 1.4, $e\mathscr{R}a^*$. Hence $(ae)^* = e$ and $a = aea^*$ so that $a\delta ae$ and consequently $(ae)\lambda = (e, a\delta)$. Thus λ is surjective. Now for any $a, b \in S$, $(a^*, a\delta) = (b^*, b\delta)$ implies $(a, b) \in \mathscr{H}^* \cap \delta$ and by [4, Proposition 2.9] this gives $a = b$. Hence λ is an isomorphism. $\square$

Now we proceed to get the structure of $\mathscr{L}^*$-unipotent semigroups which are bands of cancellative monoids as an analogue of [1, Theorem 30].

Let S be an $\mathscr{L}^*$-unipotent semigroup and E be its set of idempotents. Suppose that S is a band of cancellative monoids. Put $E/\varepsilon = \mathscr{Y}$. Note that for any $e, f \in E$, $e\delta f$ if and only if $E(e) = E(f)$, and by Corollary 1.4, $E(e)$ is a right zero semigroup. By [4, Lemma 1.5] the set of idempotents of S/δ is $\mathscr{Y} = \{e\delta \colon e \in E\}$. Further, S/δ is a semilattice $\mathscr{Y}$ of cancellative monoids. Write $S/\delta = \bigcup_{\bar{e} \in \mathscr{Y}} T_{\bar{e}}$ where $\bar{e} = e\delta$, $e \in E$ and $T_{\bar{e}} = L^*_{e\delta}(S/\delta)$. Proposition 4.5 gives that S is a spined product of E and S/δ and so we may write $S = \{(s^*, s\delta) \colon s \in S\}$. Therefore $(e, a\delta) \in S$ if and only if $(e, a\delta) \in E(e) \times T_{\bar{e}}$. It follows that $S = \bigcup_{\bar{e} \in \mathscr{Y}} (E(e) \times T_{\bar{e}})$.

For any $\bar{e}, \bar{f} \in \mathscr{Y}$ such that $\bar{e} > \bar{f}$, define $\pi_{\bar{e},\bar{f}} \colon T_{\bar{e}} \to T_{\bar{f}}$ by $a\pi_{e,\bar{f}} = a\bar{f}$. Clearly $a\mathscr{L}^*(S/\delta)\bar{e}$ so that $a\bar{f}\mathscr{L}^*(S/\delta)\bar{f}$, that is $a\bar{f} \in T_{\bar{f}}$ and for any $a, b \in T_{\bar{e}}$,

$$(ab)\pi_{\bar{e},\bar{f}} = ab\bar{f} = a\bar{f}b\bar{f} = a\pi_{\bar{e},\bar{f}}b\pi_{\bar{e},\bar{f}}.$$

If $\bar{e} > \bar{f} > \bar{g}$, then for any $a\mathscr{L}^*(S/\delta)\bar{e}$,

$$a\pi_{\bar{e},\bar{g}} = a\bar{g} = a\bar{f}\bar{g} = a\bar{f}\pi_{\bar{f},\bar{g}} = a\pi_{\bar{e},\bar{f}}\pi_{\bar{f},\bar{g}}.$$

Notice that for any $(u,a)\in E(e)\times T_{\bar e}$, $(v,b)\in E(f)\times T_{\bar f}$,

$$(u,a)(v,b)=(uv,ab)=(uv,ab\overline{ef})$$
$$=(uv,a\overline{ef}b\overline{ef})=(uv,a\pi_{\bar e,\overline{ef}}b\pi_{\bar f,\overline{ef}}).$$

Thus we have proved the converse part of the following theorem:

Theorem 4.6. *Let E be a band and $E=\bigcup_{\alpha\in\mathscr{Y}}E_\alpha$ be its maximal semilattice decomposition. Suppose that for each $\alpha\in\mathscr{Y}$, E_α is a right zero semigroup and to each $\alpha\in\mathscr{Y}$ assign a cancellative monoid M_α such that $M_\alpha\cap M_\beta=\emptyset$ if $\alpha\neq\beta$. Further, suppose that for $\alpha>\beta$ there exists a homomorphian*

$$\pi_{\alpha,\beta}:M_\alpha\to M_\beta$$

such that if $\alpha>\beta>\gamma$ then $\pi_{\alpha,\gamma}=\pi_{\alpha,\beta}\cdot\pi_{\beta,\gamma}$. Set $\pi_{\alpha,\alpha}$ equal to the identity automorphism on M_α. Let $S=\bigcup_{\alpha\in\mathscr{Y}}(E_\alpha\times M_\alpha)$ and define a multiplication on S by $(e,x)(f,y)=(ef,x\pi_{\alpha,\alpha\beta}\cdot y\pi_{\beta,\alpha\beta})$ for any $(e,x)\in E_\alpha\times M_\alpha$, $(f,y)\in E_\beta\times M_\beta$. Then S is an $\mathscr{L}^*$-unipotent semigroup which is a band of cancellative monoids. Conversely, any $\mathscr{L}^*$-unipotent semigroup which is a band of cancellative monoids can be constructed in this manner.

Proof. It is easy to see that S is a semigroup and that its set of idempotents

$$\{(e,1_\alpha):\ e\in E_\alpha,\ 1_\alpha \text{ is the identity of } M_\alpha,\ \alpha\in\mathscr{Y}\}$$

is a semigroup. If $(e,a)\in E_\alpha\times M_\alpha$, then a straightforward calculation shows that $(e,a)\mathscr{H}^*(e,1_\alpha)$ so that S is superabundant. It is also easy to check that S is $\mathscr{L}^*$-unipotent. Finally, since the $\mathscr{H}^*$-classes are the sets $E_\alpha\times M_\alpha$, it is clear that $\mathscr{H}^*$ is a congruence so that by Lemma 4.1, S is a band of cancellative monoids. □

We retain the notation of Theorem 4.6 and write $S=(\mathscr{Y},E_\alpha,M_\alpha,\pi_{\alpha,\beta})$ for the semigroup constructed there. Let $T=(\mathscr{X};F_\alpha,N_\alpha,\tau_{\alpha,\beta})$ be another $\mathscr{L}^*$-unipotent band of cancellative monoids. Put $E=\bigcup_{\alpha\in\mathscr{Y}}E_\alpha$, $F=\bigcup_{\alpha\in\mathscr{X}}F_\alpha$, $M=\bigcup_{\alpha\in\mathscr{Y}}M_\alpha=S/\delta$, $N=\bigcup_{\alpha\in\mathscr{X}}N_\alpha=T/\delta$. The following corollary is essentially the same as [11, Theorem 3.3]:

Corollary 4.7. *If ϱ is an isomorphism of E onto F and θ is an isomorphism of M onto N such that the diagram*

$$\begin{array}{ccccc}
M & \xrightarrow{\eta_M} & E/\varepsilon & \longleftarrow & E\\
{\scriptstyle\theta}\downarrow & & \downarrow & & \downarrow{\scriptstyle\varrho}\\
N & \xrightarrow[\eta_N]{} & F/\varepsilon & \longleftarrow & F
\end{array}$$

is commutative, then the function χ defined by $(e,a)\chi=(e\varrho,a\theta)$ for any $(e,a)\in S$ is an isomorphism of S onto T. Conversely, every isomorphism of S onto T can be expressed in this way.

Proof. Let χ be as defined above. Then, for any $(e,t)\in E\times M$,

$$(e,t)\in S \quad\Leftrightarrow\quad E(e)=t\eta_M \quad\Leftrightarrow\quad E(e\varrho)=t\theta\eta_N \quad\Leftrightarrow\quad (e\varrho,t\theta)\in T$$

and it is clear that θ is an isomorphism of S onto T.

Let χ be an isomorphism of S onto T and define ϱ,θ by putting $(e,t)\chi=(e\varrho,t\theta)$.

To see ϱ is well defined, note that if $(e,u),(e,v)\in S$, then $(e,u)\mathscr{L}^*(e,v)$ so that $(e,u)\chi\mathscr{L}^*(e,v)\chi$. Hence the first coordinate of $(e,u)\chi$ is the same as that of $(e,v)\chi$. To see that θ is well defined, note that if $(e,t),(f,t)\in S$, then $E(e)=t\eta_M=E(f)$. Hence $(e,t)\delta_S(f,t)$ so that $(e,t)\chi\delta_T(f,t)\chi$. It follows that the second coordinate of $(e,t)\chi$ is the same as that of $(f,t)\chi$.

Thus we have mappings $\tau:E\to F$ and $\theta:M\to N$. It is clear that these mappings are homomorphisms.

Let $f\in F$. Since $\eta_N:N\to F/\varepsilon$ is onto (see the proof of Proposition 4.5), there is an element $t\in N$ such that $t\eta_N=E(f)$ and $(f,t)\in T$. Let $(e,s)\in S$ be such that $(e,s)\chi=(f,t)$. Then $e\varrho=f$, so that ϱ maps E onto F. Similarly, θ maps M onto N.

Suppose next that $e\varrho=f\varrho$ where $e\in E_\alpha$, $f\in E_\beta$. Then $(e,1_\alpha),(f,1_\beta)\in S$ and $(e,1_\alpha)\chi=(e\varrho,1_\alpha\theta)\in T$, $(f\varrho,1_\beta\theta)\in T$. Since $e\varrho=f\varrho$, it follows that $1_\alpha\theta$, $1_\beta\theta$ are in the same cancellative monoid and as they are idempotents, we must have $1_\alpha\theta=1_\beta\theta$. Consequently

$$(e,1_\alpha)\chi=(e\varrho,1_\alpha\theta)=(f\varrho,1_\beta\theta)=(f,1_\beta)\chi$$

which implies $e=f$ proving that ϱ is one-to-one.

Suppose that $u\theta=v\theta$ for some $u,v\in M$, say, $u\in M_\alpha$, $v\in M_\beta$ and let $(e,u),(f,v)\in S$. Then $e\in E_\alpha$, $f\in E_\beta$. Now $(e\varrho,u\theta),(f\varrho,v\theta)\in T$ and if $u\theta=v\theta\in N_\gamma$, then $e\varrho,f\varrho\in F_\gamma$. Since ϱ is an isomorphism, it follows that $\alpha=\beta$. Thus $(e,v)\in S$,

$$(e,u)\chi=(e\varrho,u\theta)=(e\varrho,v\theta)=(e,v)\chi$$

and $u=v$. Therefore θ is one-to-one.

The commutativity of the diagram follows from the fact that for any $(e,a)\in E\times M$, we have:

$$E(e)=a_{\eta_M} \quad\Leftrightarrow\quad (e,a)\in S \quad\Leftrightarrow\quad (e,a)\chi\in T$$
$$\Leftrightarrow\quad (e\varrho,a\theta)\in T \quad\Leftrightarrow\quad E(e\varrho)=a\theta\eta_N. \qquad \square$$

The proof of the following corollary, is the same as that in [1, Theorem 31]:

Corollary 4.8. *S is isomorphic to T if and only if*

(i) *there is an isomorphism γ of $\mathscr{Y}$ onto $\mathscr{X}$,*

(ii) *there is an isomorphism ϱ of E onto F, and such that for each $\alpha\in\mathscr{Y}$, $E_\alpha\varrho\subseteq F_{\alpha\gamma}$, and*

(iii) *for each* $\alpha \in \mathscr{Y}$ *there exists an isomorphism* λ_α *of* M_α *onto* $N_{\alpha\gamma}$ *such that if* $\alpha > \beta$, *then* $\pi_{\alpha,\beta} \cdot \lambda_\beta = \lambda_\alpha \cdot \tau_{\alpha\gamma,\beta\gamma}$. □

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